The Natural Banach Space for Version Independent Risk Measures

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Abstract

Risk measures, or coherent measures of risk are often considered on the space L^∞, and important theorems on risk measures build on that space. Other risk measures, among them the most important risk measure – the Average Value-at-Risk – are well-defined on the larger space L^1 and this seems to be the natural domain space for this risk measure. Spectral risk measures constitute a further class of risk measures of central importance, and they are often considered on some L^p space. But in many situations this is possibly unnatural, because any L^p with $p > p_0$, say, is suitable to define the spectral risk measure as well. In addition to that risk measures have also been considered on Orlicz and Zygmund spaces. So it remains for discussion and clarification, what the natural domain to consider a risk measure is?

This paper introduces a norm, which is built from the risk measure, and a Banach space, which carries the risk measure in a natural way. It is often strictly larger than its original domain, and obeys the key property that the risk measure is finite valued and continuous on that space in an elementary and natural way.

Keywords: Risk Measures, Dual Representation, Fenchel–Young inequality, Stochastic dominance

Classification: 90C15, 60B05, 62P05

1 Introduction

This paper addresses coherent measures of risk (risk measures, for short) and the natural domain (the natural space), where they can be considered. Coherent measures of risk have been introduced in the seminal paper [ADEH99] in an axiomatic way and have been investigated in a series of subsequent papers in mathematical finance since then. In the actuarial literature, however, risk measures and axiomatic treatments have been considered already earlier, for example in Denneberg ([Den90]) and in this journal by Wang et al. ([WYP97]).

We state the axioms (cf. [ADH97]) for a convex risk measure ρ, mapping $\mathbb{R}-$valued random variables into the real numbers $\mathbb{R}$ or to $+\infty$. Here, the initial axioms have been adapted to follow the interpretation of loss instead of profit – the common modification in insurance – in the usual and appropriate way.

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(M) Monotonicity: $\rho(Y_1) \leq \rho(Y_2)$ whenever $Y_1 \leq Y_2$ almost surely;

(H) Positive homogeneity: $\rho(\lambda Y) = \lambda \rho(Y)$ whenever $\lambda > 0$;

(C) Convexity: $\rho((1-\lambda) Y_0 + \lambda Y_1) \leq (1-\lambda)\rho(Y_0) + \lambda\rho(Y_1)$ for $0 \leq \lambda \leq 1$;

(T) Translation equivariance[1]: $\rho(Y + c) = \rho(Y) + c$ if $c \in \mathbb{R}$.

The main observation in this paper starts with the fact that the risk measure ρ can be associated in a natural way with a seminorm, which is a norm in important cases. It is an elementary property that the risk measure is continuous with respect to the norm introduced.

We investigate this new norm for specific risk measures, starting with spectral risk measures. It turns out that the domain, where the spectral risk measure can be defined meaningful, is always strictly larger than L^∞. The respective space is a Banach space, and we study its topology, which can be compared with L^p spaces. However, the topology always differs from the topology of an L^p space.

A risk measure ρ – being a convex function – has a convex conjugate function, and the Fenchel–Moreau theorem allows recovering the initial function, the initial risk measure ρ in our situation. The convex conjugate function involves the dual of the initial space, for this reason it is essential to understand the dual of the Banach space associated with the risk measure. The norm on the dual space measures the growth of the random variable by involving second order stochastic dominance relations.

It is elaborated moreover in this paper that a risk measure cannot be defined in a meaningful way on a space larger than L^1.

The domain and the co-domain of spectral risk measures

The axioms characterizing risk measures have been stated above without giving the domain and the co-domain precisely. Indeed, important results are well-known when considering ρ as a function on L^∞, $\rho\colon L^\infty \to \mathbb{R}$: the results include Kusuoka's representation (cf. [Kus01] and (3) below) and results on continuity. We state the following example.

Proposition 1. *Every $\mathbb{R}-$valued risk measure ρ on L^∞ is Lipschitz-continuous with norm 1, it satisfies $|\rho(Y_2) - \rho(Y_1)| \leq \|Y_2 - Y_1\|_\infty$.*

Proof. See e.g. [FS04] for a proof. □

In many situations, for example when considering the trivial risk measure $\rho(\cdot) := \mathbb{E}(\cdot)$ or the Average Value-at-Risk, the domain L^∞ is not satisfactory large enough, the domain L^1 is perhaps more natural and convenient to consider in this situation.

Depending on the domain chosen for a risk measure, the co-domain is often specified to be $\mathbb{R}$, or the extended reals $\mathbb{R} \cup \{\infty\}$, in some publications even $\mathbb{R} \cup \{\infty, -\infty\}$. In this context it should be emphasized that there is an intimate relationship between the properties *continuity* of a risk measure and its *range*, the following important result clarifies the connections:

Proposition 2. *Consider a $\mathbb{R} \cup \{\infty\}-$valued, lsc. risk measure ρ defined on L^p, $1 \leq p < \infty$, satisfying (M), (C) and (T). Suppose further that $\{\rho < \infty\}$ has a non-empty interior. Then ρ is finite valued and continuous on the entire L^p.*

[1] In an economic or monetary environment this is often called Cash invariance instead.

The proof is contained in [RS06] and in [SRD09], Proposition 6.7. The preceding discussion of the latter reference also contains the following reformulation of the statement, which is more striking: A risk measure satisfying (M), (C) and (T) is either finite valued and continuous on the *entire* L^p, or it takes the value $+\infty$ on a dense subset.

Both results suggest to consider $\mathbb{R}$ (i.e. $\mathbb{R}\setminus\{\pm\infty\}$) valued risk measures solely, because these are precisely the finite valued and continuous risk measures.

Outline of the paper: The following Section 2 introduces the associated norm and elaborates its elementary property. The subsequent section, Section 3, addresses an elementary risk measure, the spectral risk measure. This risk measure is elementary, as every version independent risk measure can be built from spectral risk measures.

A space is introduced, which we call the space of *natural domain*, which is as large as possible to carry a spectral risk measure. It is verified that the associated space is a Banach space. The new norm can be used in a natural way to extend the domain of elementary risk measures, and it is elaborated which L^p spaces the space of natural domain comprises.

This section contains moreover the remarkable result, that there is no finite valued risk measure on a space larger than L^1.

We study further the topological dual of the Banach space introduced (Section 5). It turns out the dual norm can be characterized by use of the Average Value-at-Risk, the simplest risk measure, and by second order stochastic dominance. The investigations are pushed further to more general risk measures, and an even more general Banach space to carry a general risk measure is highlighted in Section 6.

A special section is added for an unexpected representation of the spectral risk measure (Section 7), and a final discussion completes the paper in Section 8.

2 The norm associated with a risk measure

The results presented in this paper start along with the observation that a risk measure ρ induces a (semi-)norm in the following elementary way.

Definition 3. Let L be a vector space of $\mathbb{R}-$valued random variables on $(\Omega, \mathcal{F}, P)$ and $\rho:\, L \to \mathbb{R}\cup\{-\infty, \infty\}$ be a risk measure. Then

$$\|\cdot\|_\rho := \rho\left(|\cdot|\right)$$

is called *associated norm*, associated with the risk measure ρ.

If no confusion may occur we shall simply write $\|\cdot\|$ to refer to $\|\cdot\|_\rho$.

The following proposition verifies that $\|\cdot\|_\rho$ is indeed a seminorm on the appropriate vector space.

Proposition 4 (Finiteness, and the seminorm property)**.** *Let ρ be a risk measure on a vector space of $\mathbb{R}-$valued random variables. Then $\|\cdot\| = \rho\left(|\cdot|\right)$ is a seminorm on $L := \{Y :\, \rho\left(|Y|\right) < \infty\}$ and ρ is finite valued on L.*

Proof. We show first that that ρ is $\mathbb{R}-$valued on $L = \{Y :\, \rho\left(|Y|\right) < \infty\}$. For this observe that $Y \le |Y|$, and by monotonicity thus $\rho\left(Y\right) \le \rho\left(|Y|\right) = \|Y\|$. Moreover it holds that $\rho\left(0\right) = 0$ [2] and

[2]Otherwise, $\rho\left(0\right) = \rho\left(2\cdot 0\right) = 2\cdot\rho\left(0\right)$ would imply $1 = 2$, a contradiction.

thus
$$0=2\cdot\rho\left(\frac{1}{2}Y+\frac{1}{2}(-Y)\right)\le 2\left(\frac{1}{2}\rho(Y)+\frac{1}{2}\rho(-Y)\right)=\rho(Y)+\rho(-Y),$$
such that $-\rho(Y)\le\rho(-Y)$. Now $-Y\le|Y|$ and, again by monotonicity, $-\rho(Y)\le\rho(-Y)\le\rho(|Y|)=\|Y\|$. Summarizing thus $|\rho(Y)|\le\|Y\|$, such that ρ is finite valued on L.

Note that
$$\|\lambda\cdot Y\|=\rho(|\lambda Y|)=\rho(|\lambda|\cdot|Y|)=|\lambda|\cdot\rho(|Y|)=|\lambda|\cdot\|Y\|,$$
and $\|\cdot\|$ thus is positively homogeneous.

Next it follows from monotonicity, positive homogeneity and convexity that
$$\begin{aligned}\|Y_1+Y_2\|=&\rho(|Y_1+Y_2|)\le\rho(|Y_1|+|Y_2|)=2\rho\left(\frac{1}{2}|Y_1|+\frac{1}{2}|Y_2|\right)\\ &\le 2\left(\frac{1}{2}\rho(|Y_1|)+\frac{1}{2}\rho(|Y_2|)\right)=\rho(|Y_1|)+\rho(|Y_2|)\\ &=\|Y_1\|+\|Y_2\|,\end{aligned}$$
and this is the triangle inequality. □

The next proposition elaborates, that the risk measure is continuous with respect to its associated norm. This consistency result on continuity generalizes Proposition 1.

Proposition 5 (Continuity)**.** *Let ρ be a risk measure, defined on a vector space of $\mathbb{R}-$valued random variables. Then ρ is Lipschitz continuous with constant 1 with respect to the seminorm $\|\cdot\|=\rho(|\cdot|)$.*

Proof. As for continuity note that
$$\begin{aligned}\rho(Y_2)&=&&2\cdot\rho\left(\frac{1}{2}Y_1+\frac{1}{2}(Y_2-Y_1)\right)\\ &\le&&2\left(\frac{1}{2}\rho(Y_1)+\frac{1}{2}\rho(Y_2-Y_1)\right)\le\rho(Y_1)+\rho(|Y_2-Y_1|)\end{aligned}$$
by convexity and monotonicity. It follows that $\rho(Y_2)-\rho(Y_1)\le\|Y_2-Y_1\|$. Interchanging the roles of Y_1 and Y_2 reveals that
$$|\rho(Y_2)-\rho(Y_1)|\le\|Y_2-Y_1\|,$$
the assertion. To accept that the Lipschitz constant 1 cannot be improved consider the particular choices $Y_1:=0$ and $Y_2:=\mathbb{1}$ in view of translation equivariance (T). □

3 Spectral risk measures

Among the initial attempts to introduce premium principles to price insurance contracts are distorted probabilities, a concept which can be summarized nowadays by distorted acceptability functionals (cf. [PR07]) or spectral risk measures. Spectral risk measures – or the weighted Value-at-Risk (cf. [Che06]), which is a more suggestive term – have been considered for example in [AS02, Ace02]. This risk measure involves the Value-at-Risk at level p,
$$\mathsf{V@R}_p(Y):=F_Y^{-1}(p):=\inf\{y:\ P(Y\le y)\ge p\},$$

which is the left-continuous, lower semi-continuous (lsc.) *quantile*; the spectral risk measure (or weighted $\mathsf{V@R}$) then is the functional

$$\rho_\sigma(Y) := \int_0^1 \sigma(u)\, \mathsf{V@R}_u(Y)\, \mathrm{d}u, \tag{1}$$

mapping a random variable Y to a real number, if the integral exists.

The function $\sigma:\,[0,1]\to\mathbb{R}_0^+$, called the *spectrum* or *spectral function*, is a weight function. To build a reasonable premium principle the function σ should obey some properties to be consistent with the axioms imposed on risk measures: first, associating Y with loss, σ should evaluate to non-negative reals, $\mathbb{R}_0^+$. Higher losses should be weighted higher, thus σ should be non-decreasing. And finally, as σ represents a weight function, it is natural to request $\int_0^1 \sigma(u)\,\mathrm{d}u = 1$.

An important, elementary spectral risk measure satisfying all axioms above is the Average Value-at-Risk, which is specified by the spectral function

$$\sigma_\alpha(u) := \begin{cases} 0 & \text{if } u<\alpha \\ \frac{1}{1-\alpha} & \text{else,} \end{cases}$$

that is

$$\mathsf{AV@R}_\alpha(Y) := \frac{1}{1-\alpha}\int_\alpha^1 \mathsf{V@R}_u(Y)\,\mathrm{d}u \qquad (\alpha<1), \tag{2}$$

and for $\alpha=1$ the Average Value-at-Risk per definition is

$$\mathsf{AV@R}_1(Y) := \lim_{\alpha\nearrow 1}\mathsf{AV@R}_\alpha(Y) = \operatorname{ess\,sup} Y \qquad (\alpha=1).$$

The domain of spectral risk measures

It is obvious that the Average Value-at-Risk ($\alpha<1$) may be well-defined on L^1, with the result that

$$|\mathsf{AV@R}_\alpha(Y)| \le \frac{1}{1-\alpha}\mathbb{E}|Y| = \frac{1}{1-\alpha}\|Y\|_1 < \infty \qquad \left(Y\in L^1\right),$$

that means that $\mathsf{AV@R}_\alpha$ is finite valued whenever $Y\in L^1$. This is not the case, however, for $\alpha=1$: a restriction to the smaller space $L^\infty \subset L^1$ is necessary in order to ensure that $\mathsf{AV@R}_1$ is finite valued,

$$|\mathsf{AV@R}_1(Y)| \le \|Y\|_\infty < \infty \qquad (Y\in L^\infty).$$

Even more peculiarities appear when considering the spectral function $\sigma(u) := \frac{1}{2\sqrt{1-u}}$. Clearly, $\sigma\in L^q$ whenever $q<2$, but $\sigma\notin L^2$. Hölder's inequality can be employed to insure that ρ_σ is finite valued on L^p ($p>2$, $\frac{1}{q}+\frac{1}{p}=1$), because

$$|\rho_\sigma(Y)| \le \|\sigma\|_q \cdot \left(\int_0^1 F_Y^{-1}(u)^p\right)^{\frac{1}{p}} = \frac{1}{2}\left(\frac{2}{2-p}\right)^{\frac{1}{p}} \cdot \|Y\|_p\,,$$

and the constant $\frac{1}{2}\left(\frac{2}{2-p}\right)^{\frac{1}{p}}$ again exceeds every finite bound whenever p approaches 2 from below.

So what is a good space to consider ρ_σ? Any L^p ($p>2$) guarantees that ρ_σ is finite valued and continuous, but L^2 is obviously too large. The naïve choice $\bigcup_{p>2} L^p$ does not have a satisfying norm, or topology neither. (See, for different configurations, [CL08, CL09])

Further properties and importance of spectral risk measures

A well-known and essential representation of risk measures was elaborated by Kusuoka in [Kus01]. Kusuoka's result considers risk measures on L^∞ which are *version independent* (also: *law invariant*), i.e. which satisfy $\rho(Y) = \rho(Y')$ whenever Y and Y' share the same law, that is if $P(Y \le y) = P(Y' \le y)$ for every $y \in \mathbb{R}$.

Theorem 6 (Kusuoka's representation)**.** *A version independent risk measure ρ on L^∞ of an atomless probability space $(\Omega, \mathcal{F}, P)$ has the representation*

$$\rho(Y) = \sup_{\mu \in \mathscr{M}} \int_0^1 \mathsf{AV@R}_\alpha(Y)\, \mu(\mathrm{d}\alpha), \tag{3}$$

where $\mathscr{M}$ is a collection of probability measures on $[0, 1]$.

Kusuoka representation *of* a spectral risk measure. The Kusuoka representation of a spectral risk measure ρ_σ is provided by the probability measure $\mu_\sigma((a, b]) := \int_a^b \mathrm{d}\mu_\sigma(\alpha)$ on $[0, 1]$, where μ_σ is the non-decreasing function

$$\mu_\sigma(p) := (1-p)\,\sigma(p) + \int_0^p \sigma(u)\,\mathrm{d}u \quad (0 \le p \le 1), \qquad \mu_\sigma(p) := 0 \quad (p < 0), \tag{4}$$

which satisfies $\mu_\sigma(1) = 1$ and $\mathrm{d}\mu_\sigma(p) = (1-p)\,\mathrm{d}\sigma(p)$. It holds that

$$\rho_\sigma(Y) = \int_0^1 \mathsf{AV@R}_\alpha(Y)\, \mu_\sigma(\mathrm{d}\alpha), \tag{5}$$

which exposes the Kusuoka representation of a spectral risk measure (cf. [SP13]).

Kusuoka representation *by* spectral risk measures. Conversely, any measure μ (provided that $\mu(\{1\}) = 0$) of the representation (3) can be related to the function

$$\sigma_\mu(\alpha) = \int_0^\alpha \frac{1}{1-u}\, \mu(\mathrm{d}u), \tag{6}$$

and it holds that

$$\int_0^1 \mathsf{AV@R}_\alpha(Y)\, \mu(\mathrm{d}\alpha) = \int_0^1 \sigma_\mu(\alpha)\, \mathsf{V@R}_\alpha(Y)\, \mathrm{d}\alpha = \rho_{\sigma_\mu}(Y),$$

which is a spectral risk measure.

But even the requirement $\mu(\{1\}) = 0$ can be dropped: indeed, there is a set $\mathscr{S}$ of continuous (and thus bounded) spectral functions on $[0, 1]$, such that the relation

$$\rho(Y) = \sup_{\mu \in \mathscr{M}} \int_0^1 \mathsf{AV@R}_\alpha(Y)\, \mu(\mathrm{d}\alpha) = \sup_{\sigma \in \mathscr{S}} \int_0^1 \mathsf{V@R}_\alpha(Y)\, \sigma(\alpha)\, \mathrm{d}\alpha = \sup_{\sigma \in \mathscr{S}} \rho_\sigma(Y) \tag{7}$$

holds (cf. [PP13]). This again exposes the importance of spectral risk measures, as every version independent risk measure ρ can be built from spectral risk measures by (7).

Recall that Kusuoka's representation builds on the space L^∞. But again it is not clear, if, and to which larger space this risk measure can be extended, because every σ might allow a different domain.

4 The space of natural domain, L_σ

Let σ be a non-negative, non-decreasing, integrable function with $\int_0^1 \sigma(u)\mathrm{d}u = 1$. For Y a random variable we consider the function

$$\rho_\sigma(Y) = \int_0^1 \sigma(u)\, F_Y^{-1}(u)\, \mathrm{d}u$$

already defined in (1). For $\sigma \in L^1$ (which is a minimal requirement to insure that $\int_0^1 \sigma(u)\mathrm{d}u = 1$), ρ_σ is certainly well defined for $Y \in L^\infty$, but for other random variables the integral possibly diverges. And it might diverge to $+\infty$, to $-\infty$, or be even of the indefinite form $\infty - \infty$. The following definition respects the finiteness of the spectral risk measure in view of Proposition 4.

Definition 7. The *natural domain* corresponding to a spectral risk measure ρ_σ induced by a spectral function σ is

$$L_\sigma := \left\{ Y \in L^0 : \|Y\|_\sigma < \infty \right\},$$

where

$$\|Y\|_\sigma := \rho_\sigma(|Y|).$$

Note that $|Y| \geq 0$ is positive, such that $F_{|Y|}^{-1}(\cdot) \geq 0$ is positive as well and the condition $\rho_\sigma(|Y|) < \infty$ makes perfect sense for any measurable random variable $Y \in L^0$.

Proposition 8. *$\|\cdot\|_\sigma = \rho_\sigma(|\cdot|)$ is a norm on L_σ.*

Proof. It was already shown in Proposition 4 that $\|\cdot\|_\sigma$ is a seminorm. What remains to be shown is that $\|\cdot\|_\sigma$ separates points. For this recall that σ is positive, satisfying $\int_0^1 \sigma(p)\,\mathrm{d}p = 1$, and $F_{|Y|}(\cdot)$ is an non-decreasing and positive function as well. Hence if $\int_0^1 \sigma(p)\, F_{|Y|}^{-1}(p)\,\mathrm{d}p = 0$, then $F_{|Y|}^{-1}(\cdot) \equiv 0$, that is $Y = 0$ almost everywhere. The function $\|\cdot\|_\sigma$ thus separates points in L_σ and $\|\cdot\|_\sigma$ hence is a norm. ☐

The next theorem already elaborates that the set L_σ is large enough and at least contains L^p, whenever $\sigma \in L^q$ (and the exponents are conjugate, $\frac{1}{p} + \frac{1}{q} = 1$).

Theorem 9 (Comparison with L^p). *Let σ be fixed.*

(i) If $\sigma \in L^q$ for some $q \in [1, \infty]$ with conjugate exponent p, then

$$L^\infty \subset L^p \subset L_\sigma \subset L^1$$

and

$$\|Y\|_1 \leq \|Y\|_\sigma \leq \|\sigma\|_q \cdot \|Y\|_p \tag{8}$$

whenever $Y \in L^p$.

(ii) For σ bounded (i.e. $\sigma \in L^\infty$) it holds moreover that $L_\sigma = L^1$, the norms are equivalent and satisfy

$$\|Y\|_1 \leq \|Y\|_\sigma \leq \|\sigma\|_\infty \cdot \|Y\|_1 .$$

In particular, if σ is the function being constantly 1 ($\sigma = \mathbb{1}$), then $\|Y\|_\sigma = \|Y\|_1$.

Proof. Note that $\int_0^1 \sigma(p)\,\mathrm{d}p = 1$ and $\sigma(p) \geq 0$, hence there is a $\tilde{p} \in (0,1)$ such that $\sigma(p) \leq 1$ for $p < \tilde{p}$ and $\sigma(p) \geq 1$ for $p > \tilde{p}$. Note as well that $\int_0^{\tilde{p}} 1 - \sigma(p)\,\mathrm{d}p = \int_{\tilde{p}}^1 \sigma(p) - 1\,\mathrm{d}p$. Then it follows that

$$\begin{aligned}\int_0^{\tilde{p}} (1-\sigma(p))\, F_{|Y|}^{-1}(p)\,\mathrm{d}p &\leq \int_0^{\tilde{p}} (1-\sigma(p))\, F_{|Y|}^{-1}(\tilde{p})\,\mathrm{d}p \\ &= \int_{\tilde{p}}^1 (\sigma(p)-1)\, F_{|Y|}^{-1}(\tilde{p})\,\mathrm{d}p \leq \int_{\tilde{p}}^1 (\sigma(p)-1)\, F_{|Y|}^{-1}(p)\,\mathrm{d}p,\end{aligned}$$

because $F_{|Y|}^{-1}(\cdot)$ is increasing. After rearranging thus

$$\|Y\|_1 = \mathbb{E}\,|Y| = \int_0^1 F_{|Y|}^{-1}(p)\,\mathrm{d}p \leq \int_0^1 F_{|Y|}^{-1}(p)\,\sigma(p)\,\mathrm{d}p = \rho_\sigma(|Y|) = \|Y\|_\sigma\,,$$

which is the first assertion. The inclusion $L_\sigma \subset L^1$ is immediate as well, as $\|Y\|_\sigma < \infty$ implies that $\|Y\|_1 < \infty$.

The inequality

$$\|Y\|_\sigma = \int_0^1 F_{|Y|}^{-1}(p)\,\sigma(t)\,\mathrm{d}p \leq \left(\int_0^1 \sigma(t)^q\right)^{\frac{1}{q}} \cdot \left(\int_0^1 F_{|Y|}^{-1}(t)^p\right)^{\frac{1}{p}} = \|\sigma\|_q \cdot \left(\mathbb{E}\,|Y|^p\right)^{\frac{1}{p}}$$

is Hölder's inequality. □

Remark 10. The inequality $\|Y\|_1 \leq \|Y\|_\sigma$ is a direct consequence of Chebyshev's sum inequality in its continuous form, which states that $\int_0^1 f(u)\,\mathrm{d}u \cdot \int_0^1 g(u)\,\mathrm{d}u \leq \int_0^1 f(u)\,g(u)\,\mathrm{d}u$ whenever f and g are both non-decreasing (choose $f = \sigma$ and $g = F_{|Y|}^{-1}$; cf. [HLP88]).

The following representation result is well-known for σ in an appropriate space. We extend it to L_σ, the result will be used in the sequel.

Proposition 11 (Representation of the spectral risk measure)**.** *ρ_σ has the equivalent representation* [3]

$$\rho_\sigma(Y) = \sup\left\{\mathbb{E}\,Y \cdot \sigma(U) : U \text{ is uniformly distributed}\right\} \tag{9}$$

on L_σ.

Remark 12. For the Average Value-at-Risk it holds in particular that

$$\mathsf{AV@R}_\alpha(Y) = \sup\left\{\mathbb{E}\,Y \cdot Z : \mathbb{E}\,Z = 1,\ 0 \leq Z \leq \frac{1}{1-\alpha}\right\} \tag{10}$$

in view of the spectral function (2).

Proof. Consider the random variable $Z = \sigma(U)$ for a uniformly distributed random variable U, then $P(Z \leq \sigma(\alpha)) = P(\sigma(U) \leq \sigma(\alpha)) \geq P(U \leq \alpha) = \alpha$, that is $\mathsf{V@R}_\alpha(Z) \geq \sigma(\alpha)$. But as $1 = \int_0^1 \sigma(\alpha)\,\mathrm{d}\alpha \leq \int_0^1 \mathsf{V@R}_\alpha(\sigma(U))\,\mathrm{d}\alpha = \mathbb{E}\,\sigma(U) = \int_0^1 \sigma(p)\,\mathrm{d}p = 1$ it follows that

$$\mathsf{V@R}_\alpha(Z) = \sigma(\alpha)\,.$$

[3] A random variable U is uniformly distributed if $P(U \leq u) = u$.

Now $F_Y^{-1}(\cdot)$ is an increasing function, and so is $\sigma(\cdot)$. By the Hardy–Littlewood rearrangement inequality (cf. [Hoe40] and [PR07, Proposition 1.8] for the respective rearrangement inequality, sometimes also referred to as *Hardy–Littlewood–Pólya inequality* – cf. [Dan05]) it follows thus that

$$\mathbb{E}\, Y\cdot\sigma(U)\leq\int_0^1 F_Y^{-1}(\alpha)\,\sigma(\alpha)\,\mathrm{d}\alpha.$$

However, if Y and U are coupled in a co-monotone way, then equality is attained, that is $\mathbb{E}\, Y\cdot\sigma(U)=\int_0^1 F_Y^{-1}(\alpha)\,\sigma(\alpha)\,\mathrm{d}\alpha$. This proves the statement in view of the definition of the spectral risk measure, (1). □

The next theorem demonstrates that the spaces L_σ really add something to L^p spaces, the space L_σ is *strictly larger* than L^p.

Theorem 13 (L_σ is larger than L^p)**.** *The following holds true:*

(i) Suppose that $\sigma\in L^q$ for $1\leq q<\infty$. Then the space of natural domain L_σ is strictly larger than L^p, $L^p\subsetneqq L_\sigma$ ($\frac{1}{p}+\frac{1}{q}=1$).

(ii) In particular the space of natural domain L_σ is (always) strictly larger than L^∞, $L^\infty\subsetneqq L_\sigma$ ($q=1$).

Remark 14. It should be noted that the statement of the latter theorem does not hold for $\sigma\in L^\infty$: In this situation ρ_σ is well-defined on L^1, and $L_\sigma=L^1$ by the preceding Theorem 9, (i).

Proof. To prove the first assertion assume that $\sigma\in L^q$ for $1<q<\infty$. Define the uniquely defined numbers $t_0:=0<t_1<t_2<\dots<1$, such that $\int_0^{t_n}\sigma(u)^q\mathrm{d}u=\frac{\|\sigma\|_q^q}{\zeta(p+1)}\sum_{j=1}^n\frac{1}{j^{p+1}}$ and observe that $\int_{t_{n-1}}^{t_n}\sigma(u)^q\mathrm{d}u=\frac{\|\sigma\|_q^q}{\zeta(p+1)}\frac{1}{n^{p+1}}$ [4]. Define the function

$$\tau(u):=\begin{cases}n & \text{if }\ t_{n-1}\leq u<t_n,\end{cases}$$

let U be uniformly distributed and consider the random variable

$$Y:=\sigma(U)^{q-1}\cdot\tau(U).\tag{11}$$

Note, by (9), that

$$\begin{aligned}\rho_\sigma(Y) &= \mathbb{E}\,\sigma(U)\,Y=\mathbb{E}\,\sigma(U)\,\sigma(U)^{q-1}\,\tau(U)=\mathbb{E}\,\sigma(U)^q\,\tau(U)\\ &= \int_0^1\sigma(u)^q\,\tau(u)\,\mathrm{d}u=\sum_{n=1}^\infty\int_{t_{n-1}}^{t_n}\sigma(u)^q\cdot n\,\mathrm{d}u\\ &= \frac{\|\sigma\|_q^q}{\zeta(p+1)}\sum_{n=1}^\infty\frac{n}{n^{p+1}}=\frac{\|\sigma\|_q^q}{\zeta(p+1)}\sum_{n=1}^\infty\frac{1}{n^p}=\|\sigma\|_q^q\,\frac{\zeta(p)}{\zeta(p+1)}<\infty,\end{aligned}$$

[4] $\zeta(p):=\sum_{n=1}^\infty\frac{1}{n^p}$ is Riemann's Zeta function, the series converges whenever $p>1$.

because $p>1$. Next,

$$\begin{aligned}\|Y\|_p^p &= \mathbb{E}\,|Y|^p=\int_0^1\sigma(u)^{(q-1)p}\,\tau(u)^p\,\mathrm{d}u\\ &= \int_0^1\sigma(u)^q\,\tau(u)^p\,\mathrm{d}u=\sum_{n=1}^{\infty}\int_{t_{n-1}}^{t_n}\sigma(u)^q\cdot n^p\,\mathrm{d}u\\ &= \frac{\|\sigma\|_q^q}{\zeta(p+1)}\sum_{n=1}^{\infty}\frac{n^p}{n^{p+1}}=\frac{\|\sigma\|_q^q}{\zeta(p+1)}\sum_{n=1}^{\infty}\frac{1}{n}=\infty.\end{aligned}$$

Hence, $Y\in L_\sigma$, but $Y\notin L^p$.

The second statement is actually the first statement with $q=1$, but the above proof needs a modification: To accept it define, as above, an increasing sequence of values by $t_0:=0<t_1<t_2<\dots<1$ satisfying $\int_0^{t_n}\sigma(t)\mathrm{d}t\geq 1-2^{-n}$. Note, that

$$\int_{t_{n-1}}^{t_n}\sigma(u)\mathrm{d}u\leq\int_{t_{n-1}}^{1}\sigma(u)\,\mathrm{d}u=1-\int_0^{t_{n-1}}\sigma(u)\mathrm{d}u\leq 2^{1-n}.$$

Define moreover the increasing function

$$\tau(\cdot):=\sum_{n=0}\mathbb{1}_{[t_n,\,1]}(\cdot)$$

(i.e. $\tau(t)=n$ if $t_{n-1}\leq t<t_n$) and observe that $\tau\nearrow\infty$ whenever $t\to 1$.

Now let U be a uniformly distributed random variable and set $Y:=\tau(U)$. Then

$$\begin{aligned}\rho_\sigma(Y)&=\int_0^1\sigma(u)\tau(u)\mathrm{d}u=\sum_{n=1}\int_{t_{n-1}}^{t_n}\sigma(u)\tau(u)\mathrm{d}u\\ &=\sum_{n=1}n\cdot\int_{t_{n-1}}^{t_n}\sigma(u)\mathrm{d}u\leq\sum_{n=1}n\cdot 2^{1-n}=4<\infty,\end{aligned}$$

so $Y\in L_\sigma$. But $Y\notin L^\infty$, because $P(Y\geq n)\geq 1-t_{n-1}>0$ by definition of τ. □

Remark 15. Notably the preceding proof applies for the random variable $Y=\sigma(U)^{q-1}\cdot\tau(U)^\alpha$ in (11) equally well whenever $1\leq\alpha<p$, such that L_σ is larger than L^p by an entire infinite dimensional manifold.

It was demonstrated above that the space L_σ is contained in L^1. The above inequality (8), $\|\cdot\|_1\leq\|\cdot\|_\sigma$, allows to prove an even much stronger result: a finite valued risk measure cannot be considered on a space larger than L^1. This is the content of the following theorem, which was communicated to the author by Prof. Alexander Shapiro (Georgia Tech). In brief: it does not make sense to consider risk measures on a space larger than L^1.

Theorem 16. *Let $L\subset L^0$ be a vector space collecting $\mathbb{R}-$valued random variables on $([0,1],\mathscr{B},\lambda)$ (the standard probability space equipped with its Borel sets) such that $L\supsetneq L^1$ and $|Y|\in L$, if $Y\in L$. Then there does not exist a version independent, finite valued risk measure on L.*

Proof. Suppose that $\rho : L \to \mathbb{R}$ is a version independent, and finite valued risk measure on L. Restricted to L^∞, Kusuoka's theorem (Theorem 6) applies and ρ takes the form $\rho(\cdot) = \sup_{\sigma \in \mathscr{S}} \rho_\sigma(\cdot)$. Choose $Y \in L \backslash L^1$, that is $\mathbb{E}|Y| = \infty$, or $\int_0^p F_{|Y|}^{-1}(u)\,\mathrm{d}u \to \infty$ whenever $p \to 1$.

Next, pick any $\sigma \in \mathscr{S}$. Define $Y_n := \min\{n, |Y|\}$ and observe that $\rho(Y_n) \le \rho(|Y|)$ by monotonicity. Note that $Y_n \in L^\infty$ and hence, by Kusuoka's representation, (8) and the particular choice of Y,

$$\rho(|Y|) \ge \rho(Y_n) \ge \rho_\sigma(Y_n) = \|Y_n\|_\sigma \ge \|Y_n\|_1 \ge \int_0^{P(|Y| \le n)} F_{|Y|}^{-1}(u)\,\mathrm{d}u \to \infty,$$

as $n \to \infty$. Hence, ρ is not finite valued on L. □

Theorem 17. *$(L_\sigma, \|\cdot\|_\sigma)$ is a Banach space over $\mathbb{R}$.*

Proof. It remains to be shown that $(L_\sigma, \|\cdot\|_\sigma)$ is complete. For this let $(Y_k)_k$ be a Cauchy sequence for $\|\cdot\|_\sigma$. By (8) the sequence $(Y_k)_k$ is a Cauchy sequence for $\|\cdot\|_1$ as well, and from completeness of L^1 it follows that there exists a limit $Y \in L^1$. We shall show that $Y \in L_\sigma$. It follows from convergence in L^1 that $(Y_k)_k$ converges in distribution, that is $F_{Y_k}(y) \to F_Y(y)$ for every point y where $F_{|Y|}$ is continuous and moreover $F_{|Y_k|}^{-1}(u) \to F_{|Y|}^{-1}(u)$ (cf. [vdV98, Chapter 21]). Now

$$\begin{aligned} \|Y\|_\sigma &= \rho_\sigma(|Y|) = \int_0^1 \sigma(t)\, F_{|Y|}^{-1}(t)\,\mathrm{d}t = \int_0^1 \sigma(t) \lim_{k\to\infty} F_{|Y_k|}^{-1}(t)\,\mathrm{d}t \\ &= \int_0^1 \sigma(t) \liminf_{k\to\infty} F_{|Y_k|}^{-1}(t)\,\mathrm{d}t \le \liminf_{k\to\infty} \int_0^1 \sigma(t)\, F_{|Y_k|}^{-1}(t)\,\mathrm{d}t = \liminf_{k\to\infty} \|Y_k\|_\sigma \end{aligned}$$

by Fatou's Lemma, which is applicable because $F_{|Y_k|}^{-1}(\cdot) \ge 0$.

As $(Y_k)_k$ is a Cauchy sequence one may pick $k^* \in \mathbb{N}$ such that $\|Y_k - Y_{k^*}\|_\sigma < 1$ and hence $\|Y_k\|_\sigma \le \|Y_{k^*}\|_\sigma + \|Y_k - Y_{k^*}\|_\sigma < \|Y_{k^*}\|_\sigma + 1 < \infty$ for all $k > k^*$ by the triangle inequality. The sequence $(Y_k)_k$ thus is uniformly bounded in its norm. Hence,

$$\|Y\|_\sigma \le \liminf_{k\to\infty} \|Y_k\|_\sigma \le \|Y_{k^*}\|_\sigma + 1 < \infty,$$

that is $Y \in L_\sigma$ and L_σ thus is complete. □

Example 18. Consider the spectrum $\sigma(\alpha) = \frac{1}{2\sqrt{1-\alpha}}$. It should be noted that $L_\sigma \supset \bigcup_{p>2} L^p$, and $\|\cdot\|_\sigma$ provides a reasonable norm on that set.

Restricted to L^p, for some $p > 2$, the open mapping theorem (cf. [Rud73] or [AB06]) insures that the norms are equivalent, that is there are constants C_1 and C_2 such that

$$C_1 \|Y\|_p \le \|Y\|_\sigma \le C_2 \|Y\|_p \qquad (Y \in L^p \subset L_\sigma).$$

The latter inequalities hold just for $Y \in L^p$, but not for $Y \in L_\sigma$.

Proposition 19. *Measurable, simple (step) functions are dense in L_σ, and in particular L^∞ is dense in L_σ.*

Proof. Given $Y \in L_\sigma$ and $\varepsilon > 0$, find $t_0 \in (0,1)$ such that $\int_0^{t_0} F_Y^{-1}(t)\sigma(u)\mathrm{d}u < \frac{\varepsilon}{3}$ and set $s(t) := F_Y^{-1}(t_0)$ whenever $t \le t_0$. Moreover, find $t_1 \in (0,1)$ such that $\int_{t_1}^1 F_Y^{-1}(t)\sigma(u)\mathrm{d}u < \frac{\varepsilon}{3}$ and set $s(t) := F_Y^{-1}(t_1)$ whenever $t \ge t_1$. In between, as $F_Y^{-1}(t)$ is non-decreasing on the compact $[t_0, t_1]$, there is an increasing step function $s(t)$ such that $\left|s(t) - F_Y^{-1}(t)\right| \sigma(t) < \frac{\varepsilon}{3}$. Let U be uniformly distributed and co-monotone with Y. Then it holds that $\left\|\tilde{Y} - s(U)\right\|_\sigma < \varepsilon$ by construction of the step function s. □

5 The Dual of the natural domain L_σ

Risk measures are convex and lower semi-continuous (cf. [JST06]) functions, hence they have a dual representation by involving the Fenchel–Moreau Theorem (also Legendre transformation, see below). This representation involves the dual space in a natural way, and hence it is of interest to understand the dual of the Banach space $(L_\sigma, \|\cdot\|_\sigma)$. We describe the norm of the dual and identify the dual with a subspace of L^1. The respective results are proven in this section, moreover essential properties of the dual are highlighted.

Theorem 20 (Fenchel–Moreau)**.** *Let $\mathscr{Y}$ be a Banach space and $f: \mathscr{Y} \to \mathbb{R} \cup \{\infty\}$ be convex and lower semi-continuous with $f(Y_0) < \infty$ for an $Y_0 \in \mathscr{Y}$. Then*

$$f^{**} = f,$$

where

$$f^*(Z^*) := \sup_{Y \in \mathscr{Y}} Z^*(Y) - f(Y) \quad \text{and} \quad f^{**}(Y) := \sup_{Z^* \in \mathscr{Y}^*} Z^*(Y) - f^*(Z^*).$$

Proof. cf. [Roc74]. □

Note, that a risk measure ρ_σ is not only lower semicontinuous, by Proposition 5 it is continuous with respect to the norm $\|\cdot\|_\sigma$ on the Banach space $\mathscr{Y} = (L_\sigma, \|\cdot\|_\sigma)$. By the Fenchel–Moreau theorem thus $\rho_\sigma^{**} = \rho_\sigma$. To involve it on its natural domain $\mathscr{Y} = (L_\sigma, \|\cdot\|_\sigma)$ its dual $\mathscr{Y}^* = (L_\sigma, \|\cdot\|_\sigma)^*$ has to be available.

Definition 21. For a spectral function σ and a random variable $Z \in L^1$ define the binary relation

$$Z \preccurlyeq \sigma \quad \text{iff} \quad \mathsf{AV@R}_\alpha(|Z|) \le \frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u \text{ for all } 0 \le \alpha < 1, \tag{12}$$

the gauge function (Minkowski functional)

$$\begin{aligned} \|Z\|_\sigma^* :&= \inf\left\{\eta \ge 0:\ \mathsf{AV@R}_\alpha(|Z|) \le \frac{\eta}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u \text{ for all } 0 \le \alpha < 1\right\} \\ &= \inf\{\eta \ge 0:\ |Z| \preccurlyeq \eta \cdot \sigma\} \end{aligned} \tag{13}$$

and the space $L_\sigma^* := \left\{Z \in L^0:\ \|Z\|_\sigma^* < \infty\right\}$.

It should be noted that the relation (12), which is a kind of second order stochastic dominance relation (cf. [DDGK06, DR04]), can be interpreted as a growth condition for $|Z|$, which is a condition on Z's tails: $Z \preccurlyeq \eta \cdot \sigma$ can only hold true if $|Z|$ does not grow faster towards ∞ than $\eta \cdot \sigma$.

Notice as well that

$$\|Z\|_\sigma^* \le \eta \text{ if and only if } \mathsf{AV@R}_\alpha(|Z|) \le \frac{\eta}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u \text{ for all } 0 \le \alpha < 1. \tag{14}$$

Moreover the functions $\alpha \mapsto \int_\alpha^1 \sigma(u)\mathrm{d}u$ and $\alpha \mapsto (1-\alpha)\,\mathsf{AV@R}_\alpha(|Z|)$ are both continuous functions on $[0, 1]$, so the maximum of their difference is attained in $[0, 1]$. Hence, the infimum in (13) will be attained as well at some $\eta \ge 0$.

Lemma 22. *The unit ball of the norm $\|\cdot\|_\sigma^*$ is*

$$B_\sigma := \left\{ Z \in L^1 :\ \mathsf{AV@R}_\alpha\left(|Z|\right) \leq \frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u \text{ for all } 0 \leq \alpha < 1 \right\},$$

which is an absolutely convex set.

Proof. Just observe that

$$\begin{aligned} \mathsf{AV@R}_\alpha\left(|\lambda_1 Z_1 + \lambda_2 Z_2|\right) &\leq \mathsf{AV@R}_\alpha\left(|\lambda_1 Z_1| + |\lambda_2 Z_2|\right) \\ &= 2\cdot \mathsf{AV@R}_\alpha\left(\frac{1}{2}|\lambda_1 Z_1| + \frac{1}{2}|\lambda_2 Z_2|\right) \\ &\leq |\lambda_1|\,\mathsf{AV@R}_\alpha\left(|Z_1|\right) + |\lambda_2|\,\mathsf{AV@R}_\alpha\left(|Z_2|\right) \end{aligned}$$

by monotonicity, convexity and positive homogeneity (sub-additivity). For Z_1, $Z_2 \in B_\sigma$ and $|\lambda_1| + |\lambda_2| \leq 1$ it follows thus that $\lambda_1 Z_1 + \lambda_2 Z_2 \in B_\sigma$ and B_σ is absolutely convex. □

Comparison with L^1. For $Z \in L^\sigma$, $\|Z\|_\sigma^* \leq \eta$ implies that $\mathbb{E}\,|Z| \leq \eta$ (by the choice $\alpha = 0$ in (14)), hence

$$\|Z\|_1 \leq \|Z\|_\sigma^* \tag{15}$$

and $L_\sigma^* \subset L^1$.

Comparison with L^∞. Suppose that σ is bounded and $Z \in L^\infty$. Then $\mathsf{AV@R}_\alpha\left(|Z|\right) \to \|Z\|_\infty$ and $\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u \to \|\sigma\|_\infty$, as $\alpha \to 1$, and consequently $\|Z\|_\infty \leq \eta \cdot \|\sigma\|_\infty$ has to hold by (14) for η to be feasible, that is

$$\|Z\|_\infty \leq \|Z\|_\sigma^* \cdot \|\sigma\|_\infty . \tag{16}$$

Upper bound. An upper bound for the norm $\|\cdot\|_\sigma^*$ is given by

$$\|Z\|_\sigma^* \leq \sup_{0 \leq u < 1} \frac{F_{|Z|}^{-1}(u)}{\sigma(u)},$$

where the conventions $\frac{0}{0} = 0$ and $\frac{1}{0} = \infty$ have to be employed. Indeed, if $\frac{F_{|Z|}^{-1}(u)}{\sigma(u)} \leq \eta$, then integrating gives $(1-\alpha)\,\mathsf{AV@R}_\alpha\left(|Z|\right) = \int_\alpha^1 F_{|Z|}^{-1}(u)\mathrm{d}u \leq \eta \cdot \int_\alpha^1 \sigma(u)\,\mathrm{d}u$, which in turn means that $\|Z\|_\sigma^* \leq \eta$. Notice, however, that $Z \mapsto \sup_{0\leq u<1} \frac{F_{|Z|}^{-1}(u)}{\sigma(u)}$ is not a norm, it does not satisfy the triangle inequality.

Simple functions. For $Z = \sum_{j=1}^n a_j \mathbb{1}_{A_j}$ a simple (step) function, $\alpha \mapsto (1-\alpha)\,\mathsf{AV@R}_\alpha\left(|Z|\right) = \int_0^1 F_{|Z|}^{-1}(u)\,\mathrm{d}u$ is piecewise linear. As $\alpha \mapsto \int_\alpha^1 \sigma(u)\,\mathrm{d}u$ is concave (this is, because σ is increasing), the defining condition (14) has to be verified on finite many points only, such that simple functions are contained in L_σ^*.

Proposition 23. *The pair $\left(L_\sigma^*, \|\cdot\|_\sigma^*\right)$ is a Banach space.*

Proof. Notice first that $\|Z\|_\sigma^* = 0$ implies that $\mathsf{AV@R}_\alpha\left(|Z|\right) = 0$ for all $\alpha < 1$, so

$$0 = \lim_{\alpha \nearrow 1} \mathsf{AV@R}_\alpha\left(|Z|\right) = \operatorname{ess\,sup}|Z|,$$

that is $Z=0$ almost everywhere, such that $\|\cdot\|_\sigma^*$ separates points in L_σ^*.

Positive homogeneity is immediate and inherited from the Average Value-at-Risk.

As for the triangle inequality let η_1 and η_2, resp. satisfy (13) for Z_1 and Z_2, resp.. Then, by monotonicity and sub-additivity of the Average Value-at-Risk,

$$\mathsf{AV@R}_\alpha\left(|Z_1+Z_2|\right)\le \mathsf{AV@R}_\alpha\left(|Z_1|+|Z_2|\right)\le \mathsf{AV@R}_\alpha\left(|Z_1|\right)+\mathsf{AV@R}_\alpha\left(|Z_2|\right)$$

such that

$$\mathsf{AV@R}_\alpha\left(|Z_1+Z_2|\right)\le \frac{\eta_1+\eta_2}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u,$$

that is finally $\|Z_1+Z_2\|_\sigma^*\le \|Z_1\|_\sigma^*+\|Z_2\|_\sigma^*$, the triangle inequality.

Finally completeness remains to be shown. For this let Z_k be a Cauchy sequence. Hence there is k^*, such that $\|Z_k\|_\sigma^*\le \|Z_{k^*}\|_\sigma^*+\|Z_k-Z_{k^*}\|_\sigma^*\le \|Z_{k^*}\|_\sigma^*+1$, that is there is $\eta\ge 0$ (η satisfies $\eta\le \|Z_{k^*}\|_\sigma^*+1$) such that

$$\mathsf{AV@R}_\alpha\left(|Z_k|\right)\le \frac{\eta}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u$$

for all k and $\alpha\in(0,1)$. Next, by (15) Z_k is a Cauchy sequence for L^1 as well, hence there is a limit $Z\in L^1$, and Z_k converges in distribution and in quantiles. By Fatou's inequality,

$$\begin{aligned}
(1-\alpha)\,\mathsf{AV@R}_\alpha\left(|Z|\right) &= \int_\alpha^1 F_{|Z|}^{-1}(u)\mathrm{d}u=\int_\alpha^1 \liminf_{k\to\infty} F_{|Z_k|}^{-1}(u)\mathrm{d}u\\
&\le \liminf_{k\to\infty}\int_\alpha^1 F_{|Z_k|}^{-1}(u)\mathrm{d}u=(1-\alpha)\liminf_{k\to\infty}\mathsf{AV@R}_\alpha\left(|Z_k|\right)\\
&\le \eta\cdot\int_\alpha^1 \sigma(u)\mathrm{d}u.
\end{aligned}$$

Hence, the limit $Z\in L^1$ satisfies the defining conditions to qualify for L_σ^* and $\|Z\|_\sigma^*\le\eta$. It follows that $Z\in L_\sigma^*$ and $\left(L_\sigma^*,\,\|\cdot\|_\sigma^*\right)$ thus is a Banach space. □

Theorem 24. *The space* $\left(L_\sigma^*,\,\|\cdot\|_\sigma^*\right)$ *is the dual of* $\left(L_\sigma,\,\|\cdot\|_\sigma\right)$.

Proof. Let $Y\in L_\sigma$ and $Z\in L_\sigma^*$ with $\|Z\|_\sigma^*=:\eta$ be chosen. Then note that

$$|\mathbb{E}YZ|\le \mathbb{E}\,|Y|\cdot|Z|\le \int_0^1 F_{|Y|}^{-1}(u)\,F_{|Z|}^{-1}(u)\,\mathrm{d}u$$

by the Hardy–Littlewood–Pólya inequality. To abbreviate the notation we introduce the functions $S(u):=\int_u^1\sigma(p)\mathrm{d}p$ and $G(u):=\int_u^1 F_{|Z|}^{-1}(p)\,\mathrm{d}p$ (they are well defined, because $\sigma\in L^1$ and $Z\in L^1$). Then, by Riemann–Stieltjes integration by parts,

$$\begin{aligned}
\int_0^1 F_{|Y|}^{-1}(u)\,F_{|Z|}^{-1}(u)\,\mathrm{d}u &= -\int_0^1 F_{|Y|}^{-1}(u)\,\mathrm{d}G(u)\\
&= -\,F_{|Y|}^{-1}(u)\,G(u)\Big|_{u=0}^1+\int_0^1 G(u)\,\mathrm{d}F_{|Y|}^{-1}(u)\\
&= F_{|Y|}^{-1}(0)\cdot\mathbb{E}\,|Z|+\int_0^1 G(u)\,\mathrm{d}F_{|Y|}^{-1}(u)\,.
\end{aligned}$$

Now note that $F_{|Y|}^{-1}(\cdot)$ is an increasing function, and $G(u)=\int_u^1 F_{|Z|}^{-1}(p)\,\mathrm{d}p\le\eta\cdot\int_u^1\sigma(p)\,\mathrm{d}p=\eta\cdot S(u)$ because $\|Z\|_\sigma^*\le\eta$. Thus, and employing again Riemann–Stieltjes integration by parts,

$$\begin{aligned}
|\mathbb{E}YZ| &\le F_{|Y|}^{-1}(0)\cdot\|Z\|_1+\eta\cdot\int_0^1 S(u)\,\mathrm{d}F_{|Y|}^{-1}(u)\\
&= F_{|Y|}^{-1}(0)\cdot\|Z\|_1+\eta\cdot S(u)\,F_{|Y|}^{-1}(u)\Big|_{u=0}^1-\eta\cdot\int_0^1 F_{|Y|}^{-1}(u)\,\mathrm{d}S(u)\\
&= F_{|Y|}^{-1}(0)\cdot\|Z\|_1-\eta\cdot F_{|Y|}^{-1}(0)+\eta\cdot\int_0^1 F_{|Y|}^{-1}(u)\,\sigma(u)\,\mathrm{d}u\\
&= F_{|Y|}^{-1}(0)\cdot(\|Z\|_1-\eta)+\eta\cdot\int_0^1 F_{|Y|}^{-1}(u)\,\sigma(u)\,\mathrm{d}u\\
&= F_{|Y|}^{-1}(0)\cdot\left(\|Z\|_1-\|Z\|_\sigma^*\right)+\|Z\|_\sigma^*\cdot\int_0^1 F_{|Y|}^{-1}(u)\,\sigma(u)\,\mathrm{d}u.
\end{aligned}$$

Finally observe that $F_{|Y|}^{-1}(0)=\operatorname{ess\,inf}|Y|\ge 0$ and $\|Z\|_1-\|Z\|_\sigma^*\le 0$ by (15), hence

$$|\mathbb{E}YZ|\le\|Z\|_\sigma^*\cdot\int_0^1 F_{|Y|}^{-1}(u)\,\sigma(u)\,\mathrm{d}u=\rho_\sigma(|Y|)\cdot\|Z\|_\sigma^*=\|Y\|_\sigma\cdot\|Z\|_\sigma^*.$$

This proves that for every $Z\in L_\sigma^*$ the linear mapping $Y\mapsto\mathbb{E}\,YZ$ is continuous.

It remains to be shown that every continuous, linear mapping ζ in the dual of L_σ ($\zeta\in(L_\sigma,\|\cdot\|_\sigma)^*$) takes the form $\zeta:Y\mapsto\mathbb{E}YZ$ for some $Z\in L_\sigma^*$. For this consider the (signed) measure $\mu(A):=\zeta(\mathbb{1}_A)$. If $A=\bigcup_{i=1}^\infty A_i$ is a disjoint union of measurable sets, then $\mathbb{1}_A=\sum_{i=1}^\infty\mathbb{1}_{A_i}$. Clearly,

$$\left\|\mathbb{1}_A-\sum_{i=1}^n\mathbb{1}_{A_i}\right\|_\sigma\le\int_{P(A)-\sum_{i=1}^n P(A_i)}^1\sigma(u)\mathrm{d}u\to 0$$

as P is sigma-finite and $\sigma\in L^1$. It follows by continuity of ζ with respect to $\|\cdot\|_\sigma$ that

$$\mu(A)=\zeta(\mathbb{1}_A)=\zeta\left(\sum_{i=1}^\infty\mathbb{1}_{A_i}\right)=\sum_{i=1}^\infty\zeta(\mathbb{1}_{A_i})=\sum_{i=1}^\infty\mu(A_i),$$

hence μ is a sigma-finite measure. If $P(A)=0$, then

$$|\zeta(\mathbb{1}_A)|\le\|\zeta\|\cdot\|\mathbb{1}_A\|_\sigma=\|\zeta\|\cdot\int_0^1\sigma(u)\,F_{\mathbb{1}_A}^{-1}(u)\,\mathrm{d}u=0,$$

because $F_{\mathbb{1}_A}^{-1}(u)=0$ for every $u<1$. It follows that $\mu(A)=\zeta(\mathbb{1}_A)=0$, such that μ is moreover absolutely continuous with respect to P.

Let Z be the Radon–Nikodým derivative, $\mathrm{d}\mu=Z\mathrm{d}P$. Then $\zeta(\mathbb{1}_A)=\mu(A)=\int_A Z\mathrm{d}P=\int Z\mathbb{1}_A\mathrm{d}P=\mathbb{E}\,Z\mathbb{1}_A$ and hence $\zeta(\phi)=\mathbb{E}\,Z\phi$ for all simple functions ϕ by linearity and $|\mathbb{E}\,Z\phi|\le\|\zeta\|\cdot\|\phi\|_\sigma$ by continuity of ζ. Choose the function $\phi:=\operatorname{sign}Z$ (a simple function) to see that $\mathbb{E}\,|Z|\le\|\zeta\|$, that is $Z\in L^1$.

Note as well that $\mathbb{E}\,|Z|\,\phi = \mathbb{E}\,Z\cdot\operatorname{sign}(Z)\,\phi \le \|\zeta\|\cdot\|\operatorname{sign}(Z)\,\phi\|_\sigma \le \|\zeta\|\cdot\|\phi\|_\sigma$, because ρ_σ is monotone and $|\operatorname{sign}(Z)\cdot\phi| \le |\phi|$. For any measurable set A (with complement A^c) thus

$$\begin{aligned}\mathbb{E}\,|Z|\,\mathbb{1}_{A^c} &\le \|\zeta\|\cdot\|\mathbb{1}_{A^c}\|_\sigma = \|\zeta\|\cdot\rho_\sigma(\mathbb{1}_{A^c})\\ &= \|\zeta\|\cdot\int_{P(A)}^1 \sigma(u)\mathrm{d}u,\end{aligned}$$

and hence $\mathbb{E}\,|Z|\,\frac{\mathbb{1}_{A^c}}{P(A^c)} \le \|\zeta\|\cdot\frac{1}{1-P(A)}\int_{P(A)}^1\sigma(u)\mathrm{d}u$. Taking the supremum over all sets A with $P(A)\le\alpha$ gives

$$\begin{aligned}\mathsf{AV@R}_\alpha(|Z|) &= \sup_{P(A^c)\ge 1-\alpha}\mathbb{E}\,|Z|\,\frac{\mathbb{1}_{A^c}}{P(A^c)} \le \|\zeta\|\cdot\sup_{P(A)\le\alpha}\frac{1}{1-P(A)}\int_{P(A)}^1\sigma(u)\mathrm{d}u\\ &= \frac{\|\zeta\|}{1-\alpha}\int_\alpha^1\sigma(u)\mathrm{d}u\end{aligned}$$

by (10) and because σ is increasing. It follows that $\|Z\|_\sigma^* \le \|\zeta\|$ and thus $Z\in L_\sigma^*$. This completes the proof. □

Remark 25. It should be noted that the Banach space $(L_\sigma, \|\cdot\|_\sigma)$ is not necessarily reflexive. Just consider a bounded spectrum $\sigma\in L^\infty$, for which $L_\sigma = L^1$, the norms being equivalent by Theorem 9 (ii). But L^1 is not a reflexive space.

The following statement generalizes the relations (15) and (16) for general L^q spaces. It is the dual statement to Theorem 9.

Theorem 26 (Comparison with L^q)**.** *For $\sigma\in L^q$ $(1\le q\le\infty)$ it holds that*

$$\|Z\|_q \le \|Z\|_\sigma^*\cdot\|\sigma\|_q$$

whenever $Z\in L_\sigma^*$, and thus $L_\sigma^*\subset L^q$.

Moreover,

$$\frac{\|Z\|_\infty}{\|\sigma\|_\infty} \le \|Z\|_\sigma^* \le \|Z\|_\infty$$

such that the norms $\|\cdot\|_\infty$ and $\|\cdot\|_\sigma^*$ are equivalent whenever $\sigma\in L^\infty$, and in this case $L_\sigma^* = L^\infty$.

Proof. Employing $L^p - L^q$ duality and $L_\sigma - L_\sigma^*$ duality it holds that

$$\|Z\|_q = \sup_{Y\ne 0}\frac{\mathbb{E}YZ}{\|Y\|_p} = \sup_{Y\ne 0}\frac{\|Y\|_\sigma\,\|Z\|_\sigma^*}{\|Y\|_p} \le \sup_{Y\ne 0}\frac{\|\sigma\|_q\,\|Y\|_p\,\|Z\|_\sigma^*}{\|Y\|_p} = \|\sigma\|_q\cdot\|Z\|_\sigma^*$$

by (8).

The inequality, which is missing, is given by

$$\|Z\|_\sigma^* = \sup_{Y\ne 0}\frac{\mathbb{E}YZ}{\|Y\|_\sigma} = \sup_{Y\ne 0}\frac{\|Y\|_1\,\|Z\|_\infty}{\|Y\|_\sigma} \le \sup_{Y\ne 0}\frac{\|Y\|_\sigma\,\|Z\|_\infty}{\|Y\|_\sigma} = \|Z\|_\infty\,,$$

again by (8). □

6 The natural domain space $L_{\mathscr{S}}$

Kusuoka's theorem (Theorem 6) and (7) suggest to consider risk measures of the form

$$\rho_{\mathscr{S}}(\cdot) := \sup_{\sigma\in\mathscr{S}} \rho_\sigma(\cdot).$$

To investigate this general type of risk measure we define the according norm and space first.

Definition 27. The *natural domain* of $\rho_{\mathscr{S}}$, where $\mathscr{S}$ is a collection of spectral functions, is

$$L_{\mathscr{S}} := \left\{ Y \in L^1 : \|Y\|_{\mathscr{S}} < \infty \right\},$$

where

$$\|\cdot\|_{\mathscr{S}} := \rho_{\mathscr{S}}(|\cdot|) = \sup_{\sigma\in\mathscr{S}} \rho_\sigma(|\cdot|) = \sup_{\sigma\in\mathscr{S}} \|\cdot\|_\sigma.$$

Obviously, $L_{\mathscr{S}} \subset \bigcap_{\sigma\in\mathscr{S}} L_\sigma$. In view of Theorem 9 (ii) it is obvious as well that

$$L^\infty \subset L_{\mathscr{S}} \subset L^1,$$

even more, it holds that $\|Y\|_{\mathscr{S}} \le \|Y\|_\infty$ whenever $Y \in L^\infty$, and $\|Y\|_1 \le \|Y\|_{\mathscr{S}}$, whenever $Y \in L_{\mathscr{S}}$. Further, if $\sup_{\sigma\in\mathscr{S}} \|\sigma\|_q < \infty$ is finite as well, then

$$\|Y\|_{\mathscr{S}} \le \sup_{\sigma\in\mathscr{S}} \|\sigma\|_q \cdot \|Y\|_p$$

by Theorem 9, (i).

Theorem 28. *The pair $(L_{\mathscr{S}}, \|\cdot\|_{\mathscr{S}})$ is a Banach space.*

Proof. First of all it is clear that $\|\cdot\|_{\mathscr{S}}$ is a norm on $L_{\mathscr{S}}$, as it separates points, is positive homogeneous and satisfies the triangle inequality: these properties are inherited from the spaces $(L_\sigma, \|\cdot\|_\sigma)_{\sigma\in\mathscr{S}}$.

It remains to be shown that $(L_{\mathscr{S}}, \|\cdot\|_{\mathscr{S}})$ is complete. So if $(Y_k)_k$ is a Cauchy sequence in $L_{\mathscr{S}}$, then because of $\|\cdot\|_\sigma \le \|\cdot\|_{\mathscr{S}}$ it is a Cauchy sequence in any of the spaces $(L_\sigma, \|\cdot\|_\sigma)$ and it has a limit Y there. The limit is the same for all L_σ, so $Y \in \bigcap_{\sigma\in\mathscr{S}} L_\sigma$. Following (12) it holds that

$$\|Y\|_{\mathscr{S}} = \sup_{\sigma\in\mathscr{S}} \|Y\|_\sigma \le \sup_{\sigma\in\mathscr{S}} \liminf_{k\to\infty} \|Y_k\|_\sigma \le \liminf_{k\to\infty} \sup_{\sigma\in\mathscr{S}} \|Y_k\|_\sigma = \liminf_{k\to\infty} \|Y_k\|_{\mathscr{S}}$$

by the max-min inequality. Now choose $k^* \in \mathbb{N}$ such that $\|Y_k - Y_{k^*}\|_{\mathscr{S}} < 1$ for all $k > k^*$, which is possible because the sequence is Cauchy. It follows that

$$\|Y\|_{\mathscr{S}} \le \liminf_{k\to\infty} \|Y_k\|_{\mathscr{S}} \le \|Y_{k^*}\|_{\mathscr{S}} + 1 < \infty,$$

and hence $Y \in L_{\mathscr{S}}$, that is $L_{\mathscr{S}}$ is complete. □

Theorem 29. *The risk measure $\rho_{\mathscr{S}}$ is finite valued on $L_{\mathscr{S}}$, it is moreover continuous with respect to the norm $\|\cdot\|_{\mathscr{S}}$ with Lipschitz constant 1.*

Proof. The assertion follows from the more general Proposition 5. □

Examples

We give finally two examples for which the norm $\|\cdot\|_{\mathscr{S}}$ induced by a set of spectral functions $\mathscr{S}$ coincides with the norm $\|\cdot\|_p$ on L^p. Note, that this is contrast to the space L_σ, as Theorem 13 insures that L_σ is strictly larger than L^p.

Example 30 (Higher order semideviation)**.** The $p-$semideviation risk measure for $0<\lambda\leq 1$ is

$$\rho(Y):=\mathbb{E}Y+\lambda\cdot\left\|(Y-\mathbb{E}Y)_+\right\|_p.$$

Then $L_{\mathscr{S}}=L^p$, where $\mathscr{S}$ is an appropriate spectrum to generate $\rho=\rho_{\mathscr{S}}$, and the norms $\|\cdot\|_{\mathscr{S}}$ and $\|\cdot\|_p$ are equivalent.

Proof. The generating set $\mathscr{S}$ is provided in [SP13], the higher order semideviation risk measure takes the alternative form

$$\rho(Y)=\rho_{\mathscr{S}}(Y)=\sup_{\sigma\in L^q}\left(1-\frac{\lambda}{\|\sigma\|_q}\right)\mathbb{E}Y+\frac{\lambda}{\|\sigma\|_q}\rho_\sigma(Y).$$

It is evident that $\rho_{\mathscr{S}}(|Y|)\leq\left(1-\frac{\lambda}{\|\sigma\|_q}\right)\|Y\|_1+\lambda\|Y\|_p\leq(1+\lambda)\|Y\|_p$, such that $\rho_{\mathscr{S}}$ is finite valued for $Y\in L^p$. We claim that the natural domain is $L_{\mathscr{S}}=L^p$. For this suppose that $Y\in L_{\mathscr{S}}\backslash L^p$, i.e. $\|Y\|_1<\infty$, but $\|Y\|_p=\infty$. So it holds that

$$\rho_{\mathscr{S}}(Y)\geq\lambda\cdot\sup_{\sigma\in L^q}\frac{\rho_\sigma(Y)}{\|\sigma\|_q}=\lambda\cdot\sup_{Z\in L^q}\mathbb{E}Y\frac{Z}{\|Z\|_q}=\lambda\cdot\|Y\|_p=\infty$$

by L^p-L^q duality, hence $Y\notin L_{\mathscr{S}}$ and thus $L_{\mathscr{S}}=L^p$.

It follows by the open mapping theorem that the norms are equivalent. □

Example 31. Theorem 13 states that $L_\sigma\supsetneqq L^\infty$, that is to say L_σ is strictly larger than L^∞. This is not the case any more for the space $L_{\mathscr{S}}$: for this consider just the risk measure

$$\rho(Y):=\sup_{\alpha<1}\mathsf{AV@R}_\alpha(Y)\qquad(=\operatorname{ess\,sup}Y).$$

Then $\rho(Y)<\infty$ if and only if $\operatorname{ess\,sup}Y<\infty$, that is $L_{\mathscr{S}}=L^\infty$.

7 Infimum representation of the spectral risk measure and the related norm

The spectral risk measure, as well as the norm on the natural domain space, have an additional, unexpected representation. It is a generalization of the following formula (17) for the Average Value-at-Risk, [5]

$$\mathsf{AV@R}_\alpha(Y):=\inf_{y\in\mathbb{R}}y+\frac{1}{1-\alpha}\mathbb{E}(Y-y)_+,\tag{17}$$

which is given in [Pfl00], after initial results obtained in [RU00].

[5]The positive part of x is $y_+=\max\{0,y\}$.

We present the representation in the next theorem. And perhaps it should be noted that all properties above for the spectral risk norm $\left\|\cdot\right\|_\sigma$ can be derived from the following representation equally well, by involving well-known properties of the convex conjugate function such as the infimal convolution, Fenchel–Young inequality, etc..

Theorem 32 (Representation as an infimum)**.** *For any $Y \in L_\sigma$ the spectral risk measure with spectrum σ has the representation*

$$\rho_\sigma\left(Y\right) = \inf_f \mathbb{E}\, f\left(Y\right) + \int_0^1 f^*\left(\sigma\left(p\right)\right) \mathrm{d}p, \tag{18}$$

where the infimum is among all arbitrary, measurable functions $f\colon \mathbb{R} \to \mathbb{R}$ and f^* is f's convex conjugate function, i.e. $f^*\left(y\right) := \sup_x x \cdot y - f\left(x\right)$.

The statement of the inf-representation (Theorem 32) can be formulated equivalently in the following ways.

Corollary 33. *For any $Y \in L_\sigma$ the spectral risk measure with spectrum σ allows the representations*

$$\begin{aligned}\rho_\sigma\left(Y\right) &= \inf_{f \text{ convex}} \mathbb{E}\, f\left(Y\right) + \int_0^1 f^*\left(\sigma\left(p\right)\right) \mathrm{d}p \\ &= \inf\left\{\mathbb{E}\, f\left(Y\right) : \int_0^1 f^*\left(\sigma\left(p\right)\right) \mathrm{d}p \le 0\right\},\end{aligned}$$

where the latter infimum is among arbitrary, measurable functions $f\colon \mathbb{R} \to \mathbb{R}$.

Remark 34 (Interpretation). Assuming that f solves (18) one may consider the function $f_\sigma\left(\cdot\right) := f\left(\cdot\right) + \int_0^1 f^*\left(\sigma\left(p\right)\right) \mathrm{d}p$, which is the initial function f, lifted by $\int_0^1 f^*\left(\sigma\left(p\right)\right) \mathrm{d}p$. The random variable $Y_\sigma := f_\sigma\left(Y\right)$ can be interpreted as the distortion of the random variable Y, distorted by the spectral function σ. For the distorted random variable Y_σ the initial risk measure ρ_σ reduces to the simplest risk measure, the expectation, as $\rho_\sigma\left(Y\right) = \mathbb{E} f_\sigma\left(Y\right) = \mathbb{E} Y_\sigma$.

Proof of Corollary 33. It follows from the Fenchel–Moreau theorem that the bi-conjugate function $f^{**} := \left(f^*\right)^*$ is a convex and lower semi-continuous function satisfying $f^{**} \le f$ and $f^{***} = f^*$. The infimum in (18) hence – without any loss of generality – can be restricted to *convex* functions, that is

$$\rho_\sigma\left(Y\right) = \inf_{f \text{ convex}} \mathbb{E} f\left(Y\right) + \int_0^1 f^*\left(\sigma\left(p\right)\right) \mathrm{d}p.$$

As for the second assertion notice first that clearly

$$\begin{aligned}\rho_\sigma\left(Y\right) &\le \inf\left\{\mathbb{E} f\left(Y\right) + \int_0^1 f^*\left(\sigma\left(p\right)\right) \mathrm{d}p : \int_0^1 f^*\left(\sigma\left(p\right)\right) \mathrm{d}p \le 0\right\} \\ &\le \inf\left\{\mathbb{E} f\left(Y\right) : \int_0^1 f^*\left(\sigma\left(p\right)\right) \mathrm{d}p \le 0\right\}.\end{aligned}$$

Consider $f_\alpha\left(x\right) := f(x) - \alpha$ (where α a constant and f arbitrary). It holds that $f_\alpha^*\left(y\right) = f^*(y) + \alpha$, as exposed by the auxiliary Lemma 35 in the Appendix. Hence $\int_0^1 f_\alpha^*\left(\sigma\left(p\right)\right) \mathrm{d}p = \int_0^1 f^*\left(\sigma\left(p\right)\right) \mathrm{d}p + \alpha$ and

$$\mathbb{E}\, f_\alpha\left(Y\right) + \int_0^1 f_\alpha^*\left(\sigma\left(p\right)\right) \mathrm{d}p = \mathbb{E}\, f\left(Y\right) + \int_0^1 f^*\left(\sigma\left(p\right)\right) \mathrm{d}p. \tag{19}$$

Choose $\alpha := \int_0^1 f^*(\sigma(p))\,\mathrm{d}p$ such that $\int_0^1 f_\alpha^*(\sigma(p))\,\mathrm{d}p = 0$. f_α hence is feasible for (18) with the same objective as f by (19), from which the assertion follows. ☐

Proof of Theorem 32. From the definition of the convex conjugate f^* it is immediate that

$$f^*(\sigma) \geq y \cdot \sigma - f(y)$$

for all numbers y and σ (this is often called *Fenchel–Young inequality*), hence

$$f(Y) + f^*(\sigma(U)) \geq Y \cdot \sigma(U),$$

where U is any uniformly distributed random variable, i.e. U satisfies $P(U \leq u) = u$. Taking expectations it follows that

$$\mathbb{E}f(Y) + \mathbb{E}f^*(\sigma(U)) \geq \mathbb{E}\, Y \cdot \sigma(U).$$

As U is uniformly distributed it holds that

$$\mathbb{E}f^*(\sigma(U)) = \int_0^1 f^*(\sigma(u))\,\mathrm{d}u,$$

such that

$$\mathbb{E}f(Y) + \int_0^1 f^*(\sigma(u))\,\mathrm{d}u \geq \mathbb{E}\, Y \cdot \sigma(U),$$

irrespective of the uniform random variable U. Hence, by Proposition 11,

$$\mathbb{E}f(Y) + \int_0^1 f^*(\sigma(u))\,\mathrm{d}u \geq \sup_{U \text{ uniform}} \mathbb{E}\, Y \cdot \sigma(U) = \rho_\sigma(Y),$$

establishing the inequality

$$\rho_\sigma(Y) \leq \mathbb{E}f(Y) + \int_0^1 f^*(\sigma(u))\,\mathrm{d}u.$$

As for the converse inequality consider the function

$$f_\sigma(y) := \int_0^1 F_Y^{-1}(\alpha) + \frac{1}{1-\alpha}\left(y - F_Y^{-1}(\alpha)\right)_+ \mu_\sigma(\mathrm{d}\alpha),$$

where the measure μ_σ is given by (4).

$f_\sigma(y)$ is well-defined for all y because $Y \in L^1$; $f_\sigma(y)$ is moreover increasing and convex, because $y \mapsto (y-q)_+$ is increasing and convex, and because μ_σ is positive.

Recall the formula

$$\mathsf{AV@R}_\alpha(Y) = \inf_{q \in \mathbb{R}} q + \frac{1}{1-\alpha}\mathbb{E}(Y-q)_+$$

and the fact that the infimum is attained at $q = F_Y^{-1}(\alpha)$ (cf. [Pfl00] for the general formula), providing thus the explicit form

$$\mathsf{AV@R}_\alpha(Y) = F_Y^{-1}(\alpha) + \frac{1}{1-\alpha}\mathbb{E}\left(Y - F_Y^{-1}(\alpha)\right)_+.$$

Note now that, by (5) and Fubini's Theorem,

$$\begin{aligned}
\rho_\sigma(Y) &= \int_0^1 \mathsf{AV@R}_\alpha(Y)\,\mu_\sigma(\mathrm{d}\alpha)\\
&= \int_0^1 F_Y^{-1}(\alpha) + \frac{1}{1-\alpha}\mathbb{E}\left(Y - F_Y^{-1}(\alpha)\right)_+ \mu_\sigma(\mathrm{d}\alpha)\\
&= \mathbb{E}\int_0^1 F_Y^{-1}(\alpha) + \frac{1}{1-\alpha}\left(Y - F_Y^{-1}(\alpha)\right)_+ \mu_\sigma(\mathrm{d}\alpha)\\
&= \mathbb{E}\, f_\sigma(Y)\,. \qquad (20)
\end{aligned}$$

To establish the assertion (18) it needs to be shown that $\int_0^1 f_\sigma(\sigma(u))\,\mathrm{d}u \leq 0$. For this observe first that f_σ is almost everywhere differentiable (because it is convex), with derivative

$$\begin{aligned}
f_\sigma'(y) &= \int_{\left\{\alpha:\, F_Y^{-1}(\alpha)\leq y\right\}} \frac{1}{1-\alpha}\mu_\sigma(\mathrm{d}\alpha)\\
&= \int_0^{F_Y(y)} \frac{1}{1-\alpha}\mu_\sigma(\mathrm{d}\alpha) = \sigma(F_Y(y))
\end{aligned}$$

(almost everywhere) by relation (6). Moreover $f_\sigma(\sigma(u)) = \sup_y \sigma(u)\cdot y - f_\sigma(y)$, the supremum being attained at every y satisfying $\sigma(u) = f_\sigma'(y) = \sigma(F_Y(y))$, hence at $y = F_Y^{-1}(u)$, and it follows that

$$f_\sigma(\sigma(u)) = \sigma(u)\cdot F_Y^{-1}(u) - f_\sigma\left(F_Y^{-1}(u)\right).$$

Now

$$\begin{aligned}
\int_0^1 f_\sigma(\sigma(u))\,\mathrm{d}u &= \int_0^1 \sigma(u)\cdot F_Y^{-1}(u)\,\mathrm{d}u - \int_0^1 f_\sigma\left(F_Y^{-1}(u)\right)\mathrm{d}u\\
&= \rho_\sigma(Y) - \mathbb{E} f_\sigma(Y)\,.
\end{aligned}$$

But it was established already in (20) that $\rho_\sigma(Y) = \mathbb{E} f_\sigma(Y)$, so that $\int_0^1 f_\sigma(\sigma(u))\,\mathrm{d}u = 0$. This finally proves the second inequality. $\square$

The Average Value-at-Risk is a special case of the infimum in (18). Indeed, it follows from the proof that the infimum is attained at a function of the form $f_q(y) = q + \frac{1}{1-\alpha}(y-q)_+$ with conjugate

$$f_q^*(x) = \begin{cases} -q + q\,x & \text{if } 0 \leq x \leq \frac{1}{1-\alpha}\\ \infty & \text{else.}\end{cases}$$

It follows that $\int_0^1 f_\sigma(\sigma_\alpha(x))\,\mathrm{d}x = \int_0^\alpha f_\sigma(0)\,\mathrm{d}x + \int_\alpha^1 f_\sigma\left(\frac{1}{1-\alpha}\right)\mathrm{d}x = -\alpha q + \left(-q + \frac{q}{1-\alpha}\right)(1-\alpha) = 0$, such that

$$\mathsf{AV@R}_\alpha(Y) = \inf_{q\in\mathbb{R}} \mathbb{E} f_q(Y) = \inf_q\, q + \frac{1}{1-\alpha}\mathbb{E}(Y-q)_+\,,$$

the classical result. Clearly, the infimum in (17) is in $\mathbb{R}$, a much smaller space than convex functions from $\mathbb{R}$ to $\mathbb{R}$, as required in (18).

8 Summary

In this paper we associate a norm with a risk measure in a natural way. The risk measure is continuous with respect to the associated norm. This point of view allows considering spectral risk measures on its natural domain, which is a Banach space and as large as possible. The space of natural domain is considerably larger than an accordant L^p space for spectral risk measures.

As important representation theorems, as the Fenchel–Moreau theorem, involve the dual space, we study the dual space as well. Its norm can be described by a gauge functional, and the underlying set is characterized by second order stochastic dominance constraints, which measure the pace of growth of the random variable considered.

A consequence of the results of this paper is given by the fact that finite valued risk measures cannot be defined on a space lager than L^1 in a meaningful way.

An additional, unexpected representation of the spectral risk measures involving arbitrary functions and its convex conjugate completes the outline.

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Appendix

For reference and the sake of completeness we list the following elementary result for affine linear transformations of the convex conjugate function.

Lemma 35. *The convex conjugate of the function* $g(y) := \alpha + \beta y + \gamma \cdot f(\lambda y + c)$ *for* $\gamma > 0$ *and* $\lambda \neq 0$ *is*

$$g^*(x) = -\alpha - c\,\frac{x-\beta}{\lambda} + \gamma \cdot f^*\left(\frac{x-\beta}{\lambda\gamma}\right).$$

Proof. Just observe that

$$\begin{aligned}
g^*(y) &= \sup_x \; yx - g(x)\\
&= \sup_x \; yx - \alpha - \beta x - \gamma \cdot f(\lambda x + c)\\
&= \sup_x \; y\frac{x-c}{\lambda} - \alpha - \beta\frac{x-c}{\lambda} - \gamma \cdot f(x) \qquad (21)\\
&= -\alpha - c\frac{y-\beta}{\lambda} + \sup_x \; x\frac{y-\beta}{\lambda} - \gamma \cdot f(x)\\
&= -\alpha - c\frac{y-\beta}{\lambda} + \gamma \cdot \sup_x \; x\frac{y-\beta}{\lambda\gamma} - f(x)\\
&= -\alpha - c\frac{y-\beta}{\lambda} + \gamma \cdot f^*\left(\frac{y-\beta}{\lambda\gamma}\right),
\end{aligned}$$

where we have replaced x by $\frac{x-c}{\lambda}$ in (21). ∎